MATH 227: Introduction to Complex Analysis

Spring 2017-2018, Midterm 2, Duration: 60 min. + 80 min.

Name:________________________ AUB ID:________________________

Exercise	Points	Scores
1	15	
2	25	
3	15	
4	20	
5	25	
Total	100	

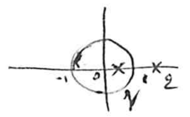
INSTRUCTIONS:

(a) Explain your answers precisely and clearly to ensure full credit.

(b) No book. No notes. No calculators.

Exercise 1. Compute $\oint_\gamma f(z)\mathrm{d}z$ where $f(z)$ and γ are given below:

(a) **(5 points)** $f(z) = \dfrac{e^{z-\frac{1}{2}}}{(z-2)(z-\frac{1}{2})}$ and $\gamma : |z| = 1$



$f(z)$ has two simple poles, $z=2$ and $z=\frac{1}{2}$, but only $z=\frac{1}{2}$ enclosed by γ

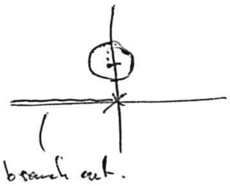
So $$\oint_{|z|=1} \frac{e^{z-\frac{1}{2}}}{(z-2)(z-\frac{1}{2})}\,dz = 2\pi i \left.\frac{e^{z-\frac{1}{2}}}{z-2}\right|_{z=\frac{1}{2}}$$

$$= 2\pi i \frac{1}{\frac{1}{2}-2}$$

$$= -\frac{4}{3}\pi i$$

(b) **(5 points)** $f(z) = \dfrac{\sinh(\mathrm{Log} z)}{\mathrm{i}z(z-\mathrm{i})}$ and $\gamma : |z-\mathrm{i}| = \frac{1}{2}$

Note that $\dfrac{\sinh(\mathrm{Log}\, z)}{iz(z-i)}$ holomorphic inside γ and ~~on~~ γ except at simple pole $z=i$



$$\oint_{|z-i|=\frac{1}{2}} f(z)\,dz = 2\pi i \left.\frac{\sinh(\mathrm{Log}\, z)}{iz}\right|_{z=i}$$

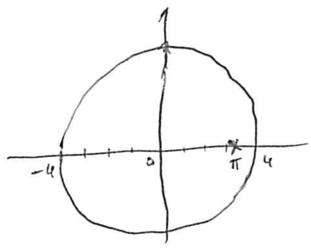
$$= 2\pi i \frac{1}{-1}\sinh(\mathrm{Log}\, i)$$

$$= -2\pi i \sinh\left(\frac{\pi}{2} i\right)$$

$$= -2\pi i \cdot i \sin\left(\frac{\pi}{2}\right)$$

$$= 2\pi$$

(c) **(5 points)** $f(z) = \dfrac{z - \sin z}{z^3(z-\pi)}$ and $\gamma : |z| = 4$



$f(z)$ has a simple pole at $z = \pi$

a removable singularity at $z = 0$

since, for instance, expanding $\dfrac{z - \sin z}{z^3} = \dfrac{1}{z^3}\left(z - \cancel{z} + \dfrac{1}{3!}z^3 - \dfrac{1}{5!}z^5 + \dots\right)$

$$= \frac{1}{3!} - \frac{1}{5!}z^2 + \dots$$

which is holomorphic at $z = 0$.

So $$\oint_{|z|=4} f(z)\,dz = 2\pi i \left.\frac{z - \sin z}{z^3}\right|_{z=\pi}$$

$$= 2\pi i \, \frac{\pi - \sin \pi}{\pi^3}$$

$$= \frac{2}{\pi} i$$

Exercise 2.

(a) **(5 points)** Compute the Laurent series expansion of $f(z) = z\cos\frac{1}{z}$ for $z \in \mathbb{C}^*$.

$$z\cos\frac{1}{z} = z \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{-2n}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{-2n+1} \quad \cdots = z - \frac{1}{2!z} + \frac{1}{4!z^3} - \cdots$$

(b) **(10 points)** Let $a, b \in \mathbb{C}$. Show that $f(z) = \dfrac{z}{(z-a)(z-b)}$ admits the Laurent series expansion

$$f(z) = \frac{a}{a-b}\frac{1}{z-a} - \frac{b}{(a-b)^2}\sum_{n=0}^{\infty}\left(\frac{z-a}{b-a}\right)^n, \quad \text{for } 0 < |z-a| < |b-a|.$$

We have,

$$f(z) = \frac{1}{(z-a)} \cdot \left[(z-a) + a\right] \cdot \frac{-1}{(b-a)-(z-a)}$$

$$= \left(1 + \frac{a}{z-a}\right) \cdot \frac{-1}{b-a} \cdot \frac{1}{1 - \frac{z-a}{b-a}}$$

$$= \left(1 + \frac{a}{z-a}\right) \cdot \left(-\frac{1}{b-a}\right) \sum_{n=0}^{\infty} \left(\frac{z-a}{b-a}\right)^n \qquad \text{for } |z-a| < |b-a|$$

$$= \sum_{n=0}^{\infty} \left(\frac{z-a}{b-a}\right)^n \left(-\frac{1}{b-a}\right) - a\sum_{n=0}^{\infty} \frac{(z-a)^{n-1}}{(b-a)^{n+1}}$$

$$= \frac{a}{a-b}\,\frac{1}{z-a} - a\sum_{n=1}^{\infty} \frac{(z-a)^{n-1}}{(b-a)^{n+1}} - \sum_{n=0}^{\infty} \frac{(z-a)^n}{(b-a)^{n+1}}$$

$$= \frac{a}{a-b}\,\frac{1}{z-a} - \frac{a}{(b-a)^2}\sum_{n=0}^{\infty} \frac{(z-a)^n}{(b-a)^n} - \frac{1}{b-a}\sum_{n=0}^{\infty} \frac{(z-a)^n}{(b-a)^n}$$

$$= \frac{a}{a-b}\,\frac{1}{z-a} - \frac{b}{(a-b)^2}\sum_{n=0}^{\infty} \left(\frac{z-a}{b-a}\right)^n$$

(c) **(10 points)** Show that

$$\cosh\left(z+\frac{1}{z}\right)=a_0+\sum_{n=1}^{\infty}a_n\left(z^n+\frac{1}{z^n}\right), \qquad \text{for all } z\in\mathbb{C}^*,$$

where

$$a_n=\frac{1}{2\pi}\int_0^{2\pi}\cos(n\theta)\cosh(2\cos\theta)\mathrm{d}\theta\,.$$

We use the definition of Laurent exp.

$$f(z)=\cosh\left(z+\frac{1}{z}\right)=\sum_{n=-\infty}^{\infty}a_n z^n \qquad \text{where } a_n=\frac{1}{2\pi i}\oint_{|w|=1}\frac{\cosh(w+w^{-1})}{w^{n+1}}\,dw$$

on $|w|=1$ $\quad w^{-1}=\bar{w}$, writing $w=e^{i\theta}$, $\quad w+w^{-1}=2\cos\theta \qquad \theta\in[0,2\pi]$

$$\frac{dw}{w}=i\,d\theta$$

So $\quad a_n=\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\cosh(2\cos\theta)\,e^{-in\theta}\,d\theta$

Now, note $f\left(\frac{1}{z}\right)=f(z)$ i.e. $\displaystyle\sum_{n=-\infty}^{\infty}a_n z^{-n}=\sum_{n=-\infty}^{\infty}a_n z^n$

i.e $\displaystyle\sum_{n=-\infty}^{\infty}a_{-n} z^{n}=\sum_{n=-\infty}^{\infty}a_n z^n$

i.e. $a_n=+a_{-n}$ by uniqueness of Laurent series exp.

So $\boxed{f(z)=a_0+\displaystyle\sum_{n=1}^{\infty}a_n\left(z^n+\frac{1}{z^n}\right)}$

Finally $a_n=a_{-n} \Rightarrow \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\cosh(2\cos\theta)\,e^{-in\theta}\,d\theta=\frac{1}{2\pi}\int_0^{2\pi}\cosh(2\cos\theta)\,e^{in\theta}\,d\theta$

i.e. a_n real

i.e. $\boxed{a_n=\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\cos(n\theta)\cosh(2\cos\theta)\,d\theta\,.}$

Rem: (not in question) $f(z)=f(-z) \Rightarrow a_{2n+1}=0 \quad \forall n\in\mathbb{Z}$

Exercise 3. **(15 points)** Let $f(z)$ be a holomorphic function with k distinct zeros z_i, $i = 1, \dots k$, of order m_i respectively. Show that $f(z) = (z - z_1)^{m_1}(z - z_2)^{m_2} \dots (z - z_k)^{m_k} g(z)$ for some holomorphic function g such that $g(z_i) \neq 0$ for all $i = 1, \dots, k$.

Since z_1 is a zero of order m_1, there exists g_1 holomorphic with $g_1(z_1) \neq 0$ s.t. $f(z) = (z-z_1)^{m_1} g_1(z)$

Now for $i \neq 1$ z_i is a zero of order m_i for $g_1(z)$ since $f(z_i) = 0 \iff \boxed{g_1(z_i) = 0}$

and $f'(z) = m_1 (z-z_1)^{m_1 - 1} g_1(z) + (z-z_1)^{m_1} g_1'(z)$

so that ~~implies that~~ $f'(z_i) = 0 \implies \boxed{g_1'(z_i) = 0}$ $\quad i \neq 1$

and $f''(z_i) = m_1(m_1 - 1)(z_i - z_1)^{m_2 - 1} g_1(z_i) + 2m_1 (z_i - z_1)^{m_1 - 1} g_1'(z_i) + (z_i - z_1)^{m_1} g_1''(z_i)$ $\quad i \neq 1$.

so that $f''(z_i) = 0 \Rightarrow \boxed{g_1''(z_i) = 0}$ $\quad i \neq 1$

and so on

$f^{(m_i - 1)}(z_i) = 0 \Rightarrow g_1^{(m_i - 1)}(z_i) = 0$ $\quad i \neq 1$

and $f^{(m_i)}(z_i) \neq 0 \Rightarrow g_1^{(m_i)}(z_i) \neq 0$ $\quad i \neq 1$

Here we have made use of the fact that $z_1 \neq z_i$ $\forall i \neq 1$

Hence z_i zero of order m_i for $g_1(z)$, $i \neq 1$. Thus, in particular, there exists g_2 holomorphic with $g_2(z_2) \neq$ such that $g_1(z) = (z-z_2)^{m_2} g_2(z)$

i.e. $f(z) = (z-z_1)^{m_1} (z-z_2)^{m_2} g_2(z)$

Repeating the argument:

Show that $g_2(z)$ has $k-2$ zeros z_i $i = 3, .., k$ of order m_i

so that $g_2(z) = (z-z_3)^{m_3} g_3(z)$

Hence, $f(z) = (z-z_1)^{m_1}(z-z_2)^{m_2} \dots (z-z_k)^{m_k} g(z)$

for some g hol. and $g(z_i) \neq 0$ $\forall i = 1 \dots k$.

Exercise 4.

(a) (**6 points**) State the maximum modulus principle and the minimum modulus principle.

Bookwork

(b) (**4 points**) Let h be a holomorphic function. By using the Cauchy-Riemann equations, show that if $|h|$ is constant, then so is h.

Let $h = u + iv$, u, v real-valued functions. CR eq. are $u_x = v_y$, $u_y = -v_x$

Now if $|h| = 0$, then $h = 0$ constant, $u = 0$, $v = 0$.

Suppose $|h| \neq 0$ so that $u^2 + v^2 \neq 0$. Then

$$|h|^2 = u^2 + v^2 = c \quad \text{constant}.$$

Differentiating yields $\quad uu_x + vv_x = 0 \quad$ and $\quad uu_y + vv_y = 0$

Multiply by u $\quad u^2u_x + uvv_x = 0 \quad$ and by v $\quad uvu_y + v^2v_y$

i.e. $\quad u^2u_x + uvu_y = 0 \quad$ i.e. $uvu_y + v^2u_x$ by CR.

adding $\Rightarrow (u^2+v^2)u_x = 0 \Rightarrow u_x = v_y = 0$. Similarly, $u_y = v_x = 0$. Hence h cst

(c) (**10 points**) Let f and g be two non-vanishing continuous functions on $\overline{\mathrm{D}}_r(0)$ that are holomorphic on $\mathrm{D}_r(0)$. Suppose $|f(z)| = |g(z)|$ for all $|z| = r$. Using (a) and (b) above, show that there exists $c \in \mathbb{C}$ with $|c| = 1$ such that $f(z) = cg(z)$ for all $z \in \overline{\mathrm{D}}_r(0)$.

Define $h(z) = \dfrac{f(z)}{g(z)}$. Then $|h(z)| = 1$ for all $|z| = r$ —— $(*)$

Since h is holomorphic and non-vanishing on $D_r(0)$, it must attain its maximum modulus and minimum modulus on $|z| = 1$. By $(*)$ they must be equal, and we have

$|h(z)| = 1$ on $\overline{D_r(0)}$ i.e. $|f(z)| = |g(z)|$ on $\overline{D_r(0)}$

By (b) we also know that $h(z) = c$ constant on $\cancel{\overline{}}\, D_r(0)$

and since $|h(z)| = 1$ $\quad |c| = 1$

Hence $f(z) = c\, g(z) \quad \forall z \in \overline{D_r(0)}$ where $|c| = 1$.

Exercise 5.

(a) (**5 points**) State Liouville's theorem.

Bookwork

(b) (**15 points**) Prove Liouville's theorem.

Bookwork

(c) (**5 points**) Let f be an entire function. Suppose there exists a positive real number M such that $\Re f(z) \leq M$ for all $z \in \mathbb{C}$. By considering $g(z) = e^{f(z)}$, prove that f is constant.

We have $|g(z)| = |e^{f(z)}| = e^{\mathrm{Re} f(z)} \leq e^{M}$ for all $z \in \mathbb{C}$.

Apply Liouville ~~so that~~ since $g(z)$ is entire bounded function, it must be constant.

Now $g'(z) = f'(z) e^{f(z)} = 0$

i.e. $f'(z) = 0$, i.e. f constant.